

ON THE RADICAL OF MULTIGRADED MODULES

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ABSTRACT. We define a functor \mathfrak{r}^* from the category of positively determined modules to the category of squarefree modules which plays the role of passing from a monomial ideal to its radical. By using this functor, we generalize several results on properties that are shared by a monomial ideal and its radical. Moreover, we study the connection of \mathfrak{r}^* to the Alexander duality and Auslander-Reiten translate functor.

INTRODUCTION

Our original motivation for this work was to generalize some of the results obtained by Herzog, Takayama, and Terai in [4]. They proved that many properties of a monomial ideal pass to its radical. It is well known that every monomial ideal I in the polynomial ring $S = K[x_1, \dots, x_n]$ over a field K is a positively \mathbf{t} -determined S -module for an appropriate $\mathbf{t} \in \mathbb{N}^n$ as it was defined by Miller in [6]. Thus, a natural way to generalize the results of [4] is to consider positively determined modules instead of monomial ideals. We show that one may define a functor \mathfrak{r}^* from the category $\mathbf{Mod}_S^{\mathbf{t}}$ of positively \mathbf{t} -determined modules to the category $\mathbf{Mod}_S^{\mathbf{1}}$ of squarefree S -modules which plays the role of passing from a monomial ideal to its radical. As it was shown in [1], to any ordering preserving map $\mathbf{q} : \mathbb{N}^n \rightarrow \mathbb{N}^n$, one may associate a functor \mathbf{q}^* from the category of \mathbb{N}^n -graded S -modules to itself which is defined as follows. For any $M \in \mathbf{Mod}_S^{\mathbb{N}^n}$ and $\mathbf{a} \in \mathbb{N}^n$, $(\mathbf{q}^*M)_{\mathbf{a}} = M_{\mathbf{q}(\mathbf{a})}$ and the S -module structure of \mathbf{q}^*M is given by the multiplication $(q^*M)_{\mathbf{b}} \xrightarrow{\mathbf{x}^{\mathbf{a}}} (q^*M)_{\mathbf{b}+\mathbf{a}}$ that maps every homogeneous element $y \in (q^*M)_{\mathbf{b}}$ to $\mathbf{x}^{q(\mathbf{a}+\mathbf{b})-q(\mathbf{b})}y$. If $f : M \rightarrow N$ is a graded morphism of \mathbb{N}^n -graded S -modules, then for every $\mathbf{a} \in \mathbb{N}^n$, the \mathbf{a} -degree component of $q^*f : q^*M \rightarrow q^*N$ is $f_{q(\mathbf{a})}$. In [1] it is shown that q^* is an exact functor. In Section 1, we have considered the following map. For $\mathbf{t} \in \mathbb{N}^n$ with $\mathbf{t} \geq \mathbf{1}$ where $\mathbf{1} = (1, \dots, 1)$, we define $\mathfrak{r} : \mathbb{N}^n \rightarrow \mathbb{N}^n$ by $\mathfrak{r}(\mathbf{a}) = (r_i(a_i))_{1 \leq i \leq n}$ where

$$r_i(a_i) = \begin{cases} t_i, & \text{if } a_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This is an ordering preserving map which induces a functor \mathfrak{r}^* which depends on \mathbf{t} from the category $\mathbf{Mod}_S^{\mathbb{N}^n}$ to itself. We showed in Section 1 that this functor

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transports the category $\mathbf{Mod}_S^{\mathbf{t}}$ into the category of squirefree modules and, moreover, for any monomial ideal $I \subset S$, we have $\mathfrak{r}^*I \cong \sqrt{I}$ as S -modules.

As it was explained in [4], the Betti numbers do not increase when one passes from a monomial ideal to its radical. We show in Theorem 1.4 that passing from a positively \mathbf{t} -determined module to its "radical" module has a similar behavior. In particular, one obtains $\text{depth } M \leq \text{depth } \mathfrak{r}^*M$ for any $M \in \mathbf{Mod}_S^{\mathbf{t}}$.

Unlike the monomial case, for a positively \mathbf{t} -determined module M , we show in Corollary 1.8, that one has only the inequality $\dim \mathfrak{r}^*M \leq \dim M$. Easy examples show that the inequality may be strict. By using the inequalities between depth and Krull dimension, we show in Theorem 1.12 that the (sequentially) Cohen-Macaulay property of M passes to the "radical" of M for any positively \mathbf{t} -determined module M with $\mathfrak{r}^*M \neq 0$.

In Section 2 we study the connection between the functor \mathfrak{r}^* and Ext . The main result of the section is Theorem 2.3 which states that for every module $M \in \mathbf{Mod}_S^{\mathbf{t}}$ there exists a natural isomorphism $\underline{\text{Ext}}_S^p(M, \omega_S)_{\geq \mathbf{0}} \cong \underline{\text{Ext}}_S^p(\mathfrak{r}^*(M), \omega_S)$ for all p , where ω_S is the canonical module of S . In particular, for any Cohen-Macaulay ideal $I \subset S$ such that S/I is a positively \mathbf{t} -determined module, it follows that the canonical module of S/\sqrt{I} is isomorphic to the positive part of $\omega_{S/I}$. From Theorem 2.3, under an additional condition, it follows that if $M \in \mathbf{Mod}_S^{\mathbf{t}}$ is a generalized Cohen-Macaulay (Buchsbaum) module, then \mathfrak{r}^*M shares the same property.

Finally, in the last two sections we show how our "radical" functor is connected to the Alexander duality (Proposition 3.1) and Auslander-Reiten translate functor (Proposition 4.2).

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1. THE \mathfrak{r}^* FUNCTOR AND FIRST APPLICATIONS

In this section we define the \mathfrak{r}^* functor on the category $\mathbf{Mod}_S^{\mathbb{N}^n}$ of the \mathbb{N}^n -graded S -modules where $S = K[x_1, \dots, x_n]$ is the polynomial ring in n variables over a field K . We first recall the basic notions and set the notation. Let \mathbb{N} be the set of non-negative integers. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ we set $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and call \mathbf{a} the degree of the monomial $\mathbf{x}^{\mathbf{a}}$. $\nu_i(u)$ denotes the exponent of variable x_i in the monomial $u \in S$. Let \leq be the partial order on \mathbb{Z}^n which is defined as follows. If $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$, then $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for $1 \leq i \leq n$. Of course, this order induces a partial order on \mathbb{N}^n .

Let $\mathbf{t} \in \mathbb{N}^n$ with $\mathbf{t} \geq \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$. According to [6], a \mathbb{Z}^n -graded S -module M is called *positively \mathbf{t} -determined* if it is finitely generated, \mathbb{N}^n -graded, and if the multiplication map $M_{\mathbf{a}} \xrightarrow{x_i} M_{\mathbf{a}+\mathbf{e}_i}$ is an isomorphism of K -vector spaces whenever $a_i \geq t_i$. Here, \mathbf{e}_i is the vector of \mathbb{Z}^n with its i -th component equal to 1 and all the others equal to 0. A monomial ideal I is positively \mathbf{t} -determined if and only if it is generated by some elements $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{t}$. Every finitely generated

\mathbb{N}^n -graded S -module is positively \mathbf{t} -determined for some $\mathbf{t} \gg \mathbf{1}$. In particular, for any 2 monomial ideals I, J with $J \supseteq I$, J/I is positively \mathbf{t} -determined for some $\mathbf{t} \gg \mathbf{1}$.

Let $\mathbf{Mod}_S^{\mathbf{t}}$ be the full subcategory of $\mathbf{Mod}_S^{\mathbb{Z}^n}$ consisting of positively \mathbf{t} -determined S -modules.

According to [1], with any order preserving map $q : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$, one may associate a functor $q^* : \mathbf{Mod}_S^{\mathbb{Z}^n} \rightarrow \mathbf{Mod}_S^{\mathbb{Z}^n}$. Since we are concerned only with \mathbb{N}^n -graded modules, that is, \mathbb{Z}^n -graded modules whose components of degree $\mathbf{a} \in \mathbb{Z}^n \setminus \mathbb{N}^n$ are all zero, we may consider the map $q : \mathbb{N}^n \rightarrow \mathbb{N}^n$.

q^* acts on modules and morphisms as follows. For a \mathbb{Z}^n -graded S -module M , the \mathbf{a} -degree component of q^*M is $(q^*M)_{\mathbf{a}} = M_{q(\mathbf{a})}$. The multiplication which gives the S -module structure of q^*M is the following. For a monomial $\mathbf{x}^{\mathbf{a}} \in S$, the map $(q^*M)_{\mathbf{b}} \xrightarrow{\cdot \mathbf{x}^{\mathbf{a}}} (q^*M)_{\mathbf{b}+\mathbf{a}}$ maps every homogeneous element $y \in (q^*M)_{\mathbf{b}}$ to $\mathbf{x}^{q(\mathbf{a}+\mathbf{b})-q(\mathbf{b})}y$.

We describe now the action of q^* on the morphisms of the category $\mathbf{Mod}_S^{\mathbb{Z}^n}$ following [1]. If $f : M \rightarrow N$ is a graded morphism of \mathbb{Z}^n -graded S -modules, then for every $\mathbf{a} \in \mathbb{Z}^n$, the \mathbf{a} -degree component of $q^*f : q^*M \rightarrow q^*N$ is $f_{q(\mathbf{a})}$. In [1] it is shown that q^* is an exact functor.

We consider now the following order preserving map. Let $\mathbf{t} \in \mathbb{N}^n, \mathbf{t} \geq \mathbf{1}$, and $\mathbf{r} : \mathbb{N}^n \rightarrow \mathbb{N}^n$ given by $\mathbf{r}(\mathbf{a}) = (r_i(a_i))_{1 \leq i \leq n}$ where

$$r_i(a_i) = \begin{cases} t_i, & \text{if } a_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that \mathbf{r} is an order preserving map, hence we may consider the functor $\mathbf{r}^* : \mathbf{Mod}_S^{\mathbb{N}^n} \rightarrow \mathbf{Mod}_S^{\mathbb{N}^n}$ associated with \mathbf{r} .

Proposition 1.1. *Let M be a positively \mathbf{t} -determined \mathbb{N}^n -graded S -module. Then \mathbf{r}^*M is a positively $\mathbf{1}$ -determined module, that is, \mathbf{r}^*M is a squarefree S -module.*

Proof. It is enough to show that, for any $\mathbf{a} \in \mathbb{N}^n$ and $i \in \text{supp}(\mathbf{a})$, the multiplication map

$$(\mathbf{r}^*M)_{\mathbf{a}} \xrightarrow{\cdot x_i} (\mathbf{r}^*M)_{\mathbf{a}+\mathbf{e}_i}$$

is an isomorphism of K -vector spaces. But this is almost obvious, since $\mathbf{x}^{\mathbf{e}_i} \cdot u = \mathbf{x}^{\mathbf{r}(\mathbf{a}+\mathbf{e}_i)-\mathbf{r}(\mathbf{a})}u = u$. The last equality is true since $\text{supp}(\mathbf{x}^{\mathbf{r}(\mathbf{a}+\mathbf{e}_i)-\mathbf{r}(\mathbf{a})}) = \emptyset$ for any $i \in \text{supp}(\mathbf{a})$. Therefore, the multiplication by x_i is the identity map, hence it is an isomorphism of vector spaces. \square

Proposition 1.2. *Let $\mathbf{t} \in \mathbb{N}^n$ with $\mathbf{t} \geq \mathbf{1}$, and let $I \subset S$ be a monomial ideal which is positively \mathbf{t} -determined, that is, for every $u \in G(I)$, $\deg(u) \leq \mathbf{t}$. Then $\mathbf{r}^*I \cong \sqrt{I}$.*

Proof. Firstly, we claim that $\mathbf{x}^{\mathbf{a}} \in \sqrt{I}$ if and only if $\mathbf{x}^{\mathbf{r}(\mathbf{a})} \in I$. Let us prove this claim. If $\mathbf{x}^{\mathbf{a}} \in \sqrt{I}$, then there exists $k \geq 1$ such that $\mathbf{x}^{k\mathbf{a}} \in I$. Obviously, we may choose k such that $ka_i \geq t_i$ for all $a_i > 0$. Then there exists $\mathbf{x}^{\mathbf{b}} \in G(I)$ such that $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{k\mathbf{a}}$, which implies that $\mathbf{b} \leq k\mathbf{a}$. Since I is positively \mathbf{t} -determined, we also have $\mathbf{b} \leq \mathbf{t}$. It then follows that $\mathbf{b} \leq \mathbf{r}(k\mathbf{a}) = \mathbf{r}(\mathbf{a})$ which implies that $\mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{r}(\mathbf{a})}$ and, therefore, $\mathbf{x}^{\mathbf{r}(\mathbf{a})} \in I$.

Conversely, let $\mathbf{x}^{\mathbf{r}(\mathbf{a})} \in I$. We obviously may find $k \geq 1$ such that $k\mathbf{a} \geq \mathbf{r}(\mathbf{a})$, hence $\mathbf{x}^{k\mathbf{a}} \in I$ and, therefore, $\mathbf{x}^{\mathbf{a}} \in \sqrt{I}$, which ends the proof of our claim.

Let $f : \sqrt{I} \rightarrow \mathfrak{r}^* I$ be the map given by $f = \oplus_{\mathbf{a}} f_{\mathbf{a}}$ where $f_{\mathbf{a}} : (\sqrt{I})_{\mathbf{a}} \rightarrow (\mathfrak{r}^* I)_{\mathbf{a}}$ is defined by $f_{\mathbf{a}}(\mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathfrak{r}(\mathbf{a})}$. The map f is obviously a graded isomorphism of K -vector spaces. We show that f is an S -module isomorphism. Indeed, for any \mathbf{a}, \mathbf{b} ,

$$f(\mathbf{x}^{\mathbf{b}} \cdot \mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathfrak{r}(\mathbf{a}+\mathbf{b})} = \mathbf{x}^{\mathfrak{r}(\mathbf{a}+\mathbf{b})-\mathfrak{r}(\mathbf{a})} \mathbf{x}^{\mathfrak{r}(\mathbf{a})} = \mathbf{x}^{\mathbf{b}} \cdot f(\mathbf{x}^{\mathbf{a}}).$$

□

In order to state the first main result, we need a preparatory lemma. Before stating it, let us set some more notation. For $\mathbf{a} \in \mathbb{N}^n$, let $S(-\mathbf{a})$ be the graded free S -module whose all graded components are obtained from those of S by shifting with the vector \mathbf{a} , and let $\sqrt{\mathbf{a}}$ be the following vector of \mathbb{N}^n :

$$(\sqrt{\mathbf{a}})_i = \begin{cases} 1, & \text{if } a_i > 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Lemma 1.3. *Let $\mathbf{t} \in \mathbb{N}^n$ with $\mathbf{t} \geq \mathbf{1}$. Then $r^*(S(-\mathbf{a})) \cong S(-\sqrt{\mathbf{a}})$ for every $\mathbf{a} \in \mathbb{N}^n$ with $\mathbf{a} \leq \mathbf{t}$.*

Proof. We obviously have the following isomorphisms:

$$\mathfrak{r}^*(S(-\mathbf{a})) \cong \mathfrak{r}^*(\mathbf{x}^{\mathbf{a}}) \cong (\mathbf{x}^{\sqrt{\mathbf{a}}}) \cong S(-\sqrt{\mathbf{a}}).$$

□

In the sequel we will always assume that $\mathfrak{r}^* M \neq 0$. Note that $\mathfrak{r}^* M = 0$ if and only if $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \in \mathbb{N}^n$ such that $a_i \in \{0, t_i\}$ for $1 \leq i \leq n$.

The following theorem shows that the graded Betti numbers go down when passing from the module to its radical. In particular, we may derive inequalities for the corresponding depths.

Theorem 1.4. *Let $\mathbf{t} \in \mathbb{N}^n$, $\mathbf{t} \geq \mathbf{1}$, and let M be a positively \mathbf{t} -determined \mathbb{N}^n -graded S -module. Then*

$$\beta_{i,\mathbf{a}}(M) \geq \beta_{i,\sqrt{\mathbf{a}}}(\mathfrak{r}^* M)$$

for all i and \mathbf{a} . In particular, the following inequality holds:

$$\text{depth } M \leq \text{depth } \mathfrak{r}^* M.$$

Proof. Let

$$\mathbb{F}_{\bullet} : \quad 0 \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}} \rightarrow \cdots \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}} \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}} \rightarrow M \rightarrow 0$$

be a minimal free resolution of M over S . Since M is positively \mathbf{t} -determined, by [6, Proposition 2.5], it follows that all the shifts in the above resolution are $\leq \mathbf{t}$. We apply \mathfrak{r}^* to \mathbb{F}_{\bullet} . By the exactness of \mathfrak{r}^* , we get a free S -resolution of $\mathfrak{r}^* M$, possibly non-minimal. Therefore, we get the inequalities between the graded Betti numbers of M and, respectively, $\mathfrak{r}^* M$. These inequalities imply that $\text{proj dim}_S M \geq \text{proj dim}_S(\mathfrak{r}^* M)$ and, by using Auslander-Buchsbaum formula, we get the inequalities between depths. □

Remark 1.5. By the above proof it follows that if

$$\mathbb{F}_\bullet : 0 \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}} \rightarrow \cdots \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}} \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}} \rightarrow M \rightarrow 0$$

is a minimal \mathbb{Z}^n -graded free resolution of M , then

$$\mathfrak{r}^* \mathbb{F}_\bullet : 0 \rightarrow \bigoplus_{\mathbf{a}} S(-\sqrt{\mathbf{a}})^{\beta_{p,\mathbf{a}}} \rightarrow \cdots \rightarrow \bigoplus_{\mathbf{a}} S(-\sqrt{\mathbf{a}})^{\beta_{1,\mathbf{a}}} \rightarrow \bigoplus_{\mathbf{a}} S(-\sqrt{\mathbf{a}})^{\beta_{0,\mathbf{a}}} \rightarrow \mathfrak{r}^* M \rightarrow 0$$

is a free resolution of $\mathfrak{r}^* M$. Moreover, if the map $\partial_i : \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \rightarrow \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{i-1,\mathbf{a}}}$ is given by the matrix $(\mathbf{x}^{\mathbf{a}_j - \mathbf{b}_k})_{j,k}$, then $\mathfrak{r}^* \partial_i : \bigoplus_{\mathbf{a}} S(-\sqrt{\mathbf{a}})^{\beta_{i,\mathbf{a}}} \rightarrow \bigoplus_{\mathbf{a}} S(-\sqrt{\mathbf{a}})^{\beta_{i-1,\mathbf{a}}}$ is given by $(\mathbf{x}^{\sqrt{\mathbf{a}_j} - \sqrt{\mathbf{b}_k}})_{j,k}$.

In the following corollary we derive some consequences of Theorem 1.4. To begin with, let $I \subset S$ be a monomial ideal. Then I is a positively \mathbf{t} -determined \mathbb{N}^n -graded S -module if we choose, for instance, $\mathbf{t} = (t_1, \dots, t_n)$ where $t_i = \max\{\nu_i(u) \mid u \in G(I)\}$. Here $G(I)$ denotes the minimal system of monomial generators of the ideal I . Proposition 1.2 says that $\mathfrak{r}^* I$ is actually \sqrt{I} . Therefore, from Theorem 1.4, we obtain as a consequence the following corollary which extends results of [4].

Corollary 1.6. *Let $I \subset J \subset S$ be monomial ideals with $\sqrt{I} \neq \sqrt{J}$. Then*

$$\beta_{i,\mathbf{a}}^S(J/I) \geq \beta_{i,\sqrt{\mathbf{a}}}^S(\sqrt{J}/\sqrt{I})$$

for all $i \geq 0$ and $\mathbf{a} \in \mathbb{N}^n$. Consequently,

$$\text{depth}_S(\sqrt{J}/\sqrt{I}) \geq \text{depth}_S(J/I).$$

In the following we study the relationship between the Krull dimension of M and $\mathfrak{r}^* M$. We first introduce the following notation. For $\mathbf{a} \in \mathbb{N}^n$, $\text{supp}(\mathbf{a}) = \{i : a_i > 0\}$ and $\text{supp}^{\mathbf{t}}(\mathbf{a}) = \{i : a_i \geq t_i\}$. We use the following convention. For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, let $\mathbf{a} \cdot \mathbf{b}$ denote the vector whose i -th component is $a_i b_i$.

Proposition 1.7. *Let M be a positively \mathbf{t} -determined S -module where $\mathbf{t} \in \mathbb{N}^n$, $\mathbf{t} \geq \mathbf{1}$. Then $\text{Ass}(\mathfrak{r}^* M) \subset \text{Ass}(M)$.*

Proof. Let $F \subset [n]$ and $P = P_F := (x_i : i \notin F)$ be an associated prime of $\mathfrak{r}^*(M)$. Then, by [9, Lemma 2.2], there exists $0 \neq u \in (\mathfrak{r}^* M)_{\mathbf{e}_F}$ such that $x_i u = 0$ for all $i \notin F$, where $\mathbf{e}_F := \sum_{i \in F} \mathbf{e}_i$. This means that there exists $0 \neq u \in M_{\mathbf{t} \cdot \mathbf{e}_F}$ such that

$$\mathbf{x}^{\mathbf{r}(\mathbf{e}_{F \cup \{i\}}) - \mathbf{r}(\mathbf{e}_i)} u = \mathbf{x}^{\mathbf{t} \cdot \mathbf{e}_i} u = x_i^{t_i} u = 0$$

for all $i \notin F$. Then we may choose a maximal monomial (with respect to divisibility) $w \in K[\{x_i : i \notin F\}]$ such that $wu \neq 0$. We claim that $P_F = \text{ann}(wu)$, which will end the proof.

If $i \notin F$, then $x_i(wu) = 0$ by the choice of w , hence $P_F \subset \text{ann}(wu)$. Let now v be a monomial in $\text{ann}(wu)$, that is, $v(wu) = 0$. Clearly, for every monomial $w' \in K[\{x_i : i \in F\}]$, we have $w'wu \neq 0$ since $\text{supp}(w') \subset \text{supp}^{\mathbf{t}}(wu)$ and M is positively \mathbf{t} -determined. This implies that there exists $i \notin F$ such that x_i divides v , thus $v \in P_F$. \square

Corollary 1.8. *Let M be a positively \mathbf{t} -determined module. Then $\dim M \geq \dim \mathfrak{r}^* M$.*

Note that the inequality $\dim \mathfrak{r}^*M \leq \dim M$ may be strict as the following example shows. On the other hand, we have $\dim \mathfrak{r}^*M = \dim M$ if and only if there exists $\mathbf{a} \in \mathbb{N}^n$ such that $\# \text{supp}^{\mathbf{t}}(\mathbf{a}) = \dim M$ and $M_{\mathfrak{r}(\mathbf{a})} \neq 0$.

Example 1.9. Let $I = (a^4d^4, a^2b^3, b^3c^2, b^3d)$ and $J = (a^3d^3, a^3b, b^2)$, $I, J \subset K[a, b, c, d]$. One may easily check that $\dim(\sqrt{J}/\sqrt{I}) = 1 < \dim(J/I) = 2$.

Let us recall that a finitely generated S -module is called *equidimensional* if all its minimal primes have the same codimension. As an immediate consequence of Proposition 1.7 we get also the following

Corollary 1.10. *Let M be a positively \mathbf{t} -determined module such that $\dim M = \dim \mathfrak{r}^*M$. If M is equidimensional, then \mathfrak{r}^*M is equidimensional, too.*

The following example shows that the implication of the above corollary is no longer true if $\dim M > \dim \mathfrak{r}^*M$.

Example 1.11. Let $P, P_1, P_2 \subset S$, $P = (x_1)$, $P_1 = (x_1, x_2)$, $P_2 = (x_1, x_3, x_4)$, and $M = (S/P)(-(1, 0, \dots, 0)) \oplus (S/P_1) \oplus (S/P_2)$. Then M is positively \mathbf{t} -determined, where $\mathbf{t} = (2, 1, \dots, 1)$, and equidimensional. We have $\mathfrak{r}^*M = (S/P_1) \oplus (S/P_2)$, thus \mathfrak{r}^*M is not equidimensional. But, of course, $\dim M > \dim \mathfrak{r}^*M$.

In [4] it is shown that if S/I is (sequentially) Cohen-Macaulay, then S/\sqrt{I} shares the same property. We are going to extend this result to any positively determined \mathbb{N}^n -graded module.

Theorem 1.12. *Let M be a positively \mathbf{t} -determined S -module with $\mathfrak{r}^*M \neq 0$ where $\mathbf{t} \in \mathbb{N}^n$, $\mathbf{t} \geq \mathbf{1}$.*

- (i) *If M is Cohen-Macaulay, then \mathfrak{r}^*M is Cohen-Macaulay and $\dim \mathfrak{r}^*M = \dim M$.*
- (ii) *If M is sequentially Cohen-Macaulay, then \mathfrak{r}^*M is sequentially Cohen-Macaulay.*

Proof. (i). By Theorem 1.4 and Corollary 1.8, we get the following inequalities:

$$\text{depth } M \leq \text{depth } \mathfrak{r}^*M \leq \dim \mathfrak{r}^*M \leq \dim M.$$

Since M is Cohen-Macaulay, we get the desired conclusions.

(ii). As M is sequentially Cohen-Macaulay, there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

by graded submodules of M such that each quotient M_i/M_{i-1} is Cohen-Macaulay and

$$\dim M_1/M_0 < \dim M_2/M_1 < \dots < \dim M_r/M_{r-1}.$$

This filtration induces the following filtration of \mathfrak{r}^*M ,

$$0 = \mathfrak{r}^*M_0 \subset \mathfrak{r}^*M_1 \subset \dots \subset \mathfrak{r}^*M_r = \mathfrak{r}^*M.$$

By (i), each factor in this filtration is either 0 or a Cohen-Macaulay module with $\dim \mathfrak{r}^*M_i/\mathfrak{r}^*M_{i-1} = \dim M_i/M_{i-1}$. By skipping the redundant factors in the above filtration we get the desired filtration for \mathfrak{r}^*M . Therefore, (ii) follows. \square

We may now derive the following corollary which extends some results of [4].

Corollary 1.13. *Let $I \subset J \subset S$ be monomial ideals such that $\sqrt{I} \neq \sqrt{J}$. Then:*

(i) *If J/I is Cohen-Macaulay, then \sqrt{J}/\sqrt{I} is Cohen-Macaulay and, moreover,*

$$\dim J/I = \dim \sqrt{J}/\sqrt{I}.$$

(ii) *If J/I is sequentially Cohen-Macaulay, then \sqrt{J}/\sqrt{I} is sequentially Cohen-Macaulay.*

2. THE FUNCTOR \mathfrak{r}^* AND Ext

For $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$ and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the \mathbb{Z}^n -graded S -module such that $M = M(\mathbf{a})$ as underlying S -modules and the degree is given by the formula $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. Following the usual convention, for $M, N \in \mathbf{Mod}_S^{\mathbb{Z}^n}$, we set $\underline{\text{Hom}}_S(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \text{Hom}_{\mathbf{Mod}_S^{\mathbb{Z}^n}}(M, N(\mathbf{a}))$. Note that if M is finitely generated, $\underline{\text{Hom}}_S(M, N) = \text{Hom}_S(M, N)$. Let $\underline{\text{Ext}}_S^i(-, N)$ (resp. $\underline{\text{Ext}}_S^i(M, -)$) be the i -th right derived functor of $\underline{\text{Hom}}_S(-, N)$ (resp. $\underline{\text{Hom}}_S(M, -)$).

In this section, we will study the relation between \mathfrak{r}^* and $\underline{\text{Ext}}$ -functor. For this purpose, we need the following three functors.

Recall that degree-shifting induces an endofunctor of $\mathbf{Mod}_S^{\mathbb{Z}^n}$. For $\mathbf{a} \in \mathbb{Z}^n$, let $\sigma_{\mathbf{a}}$ denote the functor given by shifting degree by \mathbf{a} . Thus we have $\sigma_{\mathbf{a}}(M) = M(-\mathbf{a})$ for all $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$.

For $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$ and $\mathbf{a} \in \mathbb{Z}^n$, the truncated module $\tau_{\mathbf{a}}(M) := \bigoplus_{\mathbf{b} \geq \mathbf{a}} M_{\mathbf{b}}$ is again an object of $\mathbf{Mod}_S^{\mathbb{Z}^n}$, and any morphism $M \rightarrow N$ in $\mathbf{Mod}_S^{\mathbb{Z}^n}$ induces the one $f|_{\tau_{\mathbf{a}}(M)} : \tau_{\mathbf{a}}(M) \rightarrow \tau_{\mathbf{a}}(N)$. Thus we have the functor $\tau_{\mathbf{a}} : \mathbf{Mod}_S^{\mathbb{Z}^n} \rightarrow \mathbf{Mod}_S^{\mathbb{Z}^n}$.

Let $\beta : \mathbb{N}^n \rightarrow \mathbb{Z}^n$ be the function defined by $\beta(\mathbf{a}) = (s_i(a_i))_{1 \leq i \leq n}$ where

$$s_i(a_i) = \begin{cases} t_i & \text{if } a_i \geq 1, \\ t_i - 1 & \text{otherwise.} \end{cases}$$

The induced functor $\beta^* : \mathbf{Mod}_S^{\mathbb{Z}^n} \rightarrow \mathbf{Mod}_S^{\mathbb{N}^n}$, if restricted to $\mathbf{Mod}_S^{\mathbf{t}}$, is an endofunctor of $\mathbf{Mod}_S^{\mathbf{t}}$.

For $M \in \mathbf{Mod}_S^{\mathbf{t}}$ and $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{a} \geq \mathbf{0}$, the multiplication

$$x^{\mathbf{a} \cdot \mathbf{t} - \mathbf{r}(\mathbf{a})} : M_{\mathbf{r}(\mathbf{a})} \rightarrow M_{\mathbf{a} \cdot \mathbf{t}}$$

is a K -linear isomorphism since $\text{supp}(\mathbf{a} \cdot \mathbf{t} - \mathbf{r}(\mathbf{a})) \subset \text{supp}^{\mathbf{t}} \mathbf{r}(\mathbf{a})$. Let $\phi_{\mathbf{a}}^M$ denote this map.

Now we are ready to define the two natural transformations $\Phi^{\mathbf{r}} : \text{id}_{\mathbf{Mod}_S^{\mathbf{t}}} \Longrightarrow \mathfrak{r}^*$ between endofunctors of $\mathbf{Mod}_S^{\mathbf{t}}$ and $\Psi : \beta^* \Longrightarrow \sigma_{\mathbf{1}-\mathbf{t}}$ between those of $\mathbf{Mod}_S^{\mathbb{N}^n}$. For $M \in \mathbf{Mod}_S^{\mathbf{t}}$, let $\Phi_M : M \rightarrow \mathfrak{r}^*(M)$ be the map defined as follows; for a homogeneous $u \in M_{\mathbf{a}}$ with $\mathbf{a} \geq \mathbf{0}$,

$$\Phi_M(u) := (\phi_{\mathbf{a}}^M)^{-1}(x^{\mathbf{a} \cdot (\mathbf{t}-\mathbf{1})}u) \in \mathfrak{r}^*(M)_{\mathbf{a}}.$$

For $\mathbf{a} \in \mathbb{N}^n$, it is easy to verify that $\mathbf{a} + \mathbf{t} - \mathbf{1} \geq \beta(\mathbf{a})$. For $M \in \mathbf{Mod}_S^{\mathbb{N}^n}$, we define the map $\Psi_M : \beta^*(M) \rightarrow \sigma_{\mathbf{1}-\mathbf{t}}(M)$ as follows; for $\mathbf{a} \in \mathbb{N}^n$ and a homogeneous $u \in \beta^*(M)_{\mathbf{a}}$ with $\mathbf{a} \geq \mathbf{0}$,

$$\Psi_M(u) = x^{\mathbf{a} + \mathbf{t} - \mathbf{1} - \beta(\mathbf{a})} \cdot u \in \sigma_{\mathbf{1}-\mathbf{t}}(M)_{\mathbf{a}}.$$

Lemma 2.1. *The following statements hold.*

- (1) *The above Φ is indeed a natural transformation from $\text{id}_{\mathbf{Mod}_S^t}$ to \mathfrak{r}^* .*
- (2) *Let $\iota : \mathbf{Mod}_S^1 \rightarrow \mathbf{Mod}_S^t$ be the canonical embedding. Then Φ induces the natural isomorphism $\text{id}_{\mathbf{Mod}_S^1} \implies \mathfrak{r}^* \circ \iota$ between endofunctors of \mathbf{Mod}_S^1 .*
- (3) *The above Ψ is indeed a natural transformation from \mathfrak{B}^* to σ_{1-t} .*

Proof. We will prove only the assertions (1) and (2). The rest is proved by the way similar to (1).

(1) First, we must verify that Φ_M is an S -linear map. Take any $u \in M_{\mathbf{a}}$ and $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{b} \geq \mathbf{0}$. Then

$$\begin{aligned} \phi_{\mathbf{a}+\mathbf{b}}^M(x^{\mathbf{b}} \cdot \Phi_M(u)) &= x^{(\mathbf{a}+\mathbf{b}) \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a}+\mathbf{b})} (x^{\mathfrak{r}(\mathbf{a}+\mathbf{b}) - \mathfrak{r}(\mathbf{a})} (\phi_{\mathbf{a}}^M)^{-1} (x^{\mathbf{a} \cdot (\mathbf{t}-1)} u)) \\ &= x^{(\mathbf{a}+\mathbf{b}) \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a})} (\phi_{\mathbf{a}}^M)^{-1} (x^{\mathbf{a} \cdot (\mathbf{t}-1)} u) \\ &= x^{\mathbf{b} \cdot \mathbf{t}} (x^{\mathbf{a} \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a})} (\phi_{\mathbf{a}}^M)^{-1} (x^{\mathbf{a} \cdot (\mathbf{t}-1)} u)) = x^{\mathbf{b} \cdot \mathbf{t}} \cdot x^{\mathbf{a} \cdot (\mathbf{t}-1)} u \\ &= x^{(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{t}-1)} (x^{\mathbf{b}} u), \end{aligned}$$

and hence it follows that

$$x^{\mathbf{b}} \cdot \Phi_M(u) = (\phi_{\mathbf{a}+\mathbf{b}}^M)^{-1} (x^{(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{t}-1)} (x^{\mathbf{b}} u)) = \Phi_M(x^{\mathbf{b}} u).$$

Thus Φ_M is indeed S -linear.

Next let $f : M \rightarrow N$ be a morphism in \mathbf{Mod}_S^t . We will show that the following diagram commutes;

$$\begin{array}{ccc} M & \xrightarrow{\Phi_M} & \mathfrak{r}^*(M) \\ f \downarrow & & \downarrow \mathfrak{r}^*(f) \\ N & \xrightarrow[\Phi_N]{} & \mathfrak{r}^*(N). \end{array}$$

Let $u \in M_{\mathbf{a}}$. Then

$$\begin{aligned} \phi_{\mathbf{a}}^N(\mathfrak{r}^*(f) \circ \Phi_M(u)) &= x^{\mathbf{a} \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a})} \cdot f(\Phi_M(u)) \\ &= f(x^{\mathbf{a} \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a})} \cdot \Phi_M(u)) \\ &= f(x^{\mathbf{a} \cdot \mathbf{t} - \mathfrak{r}(\mathbf{a})} \cdot (\phi_{\mathbf{a}}^M)^{-1} (x^{\mathbf{a} \cdot (\mathbf{t}-1)} u)) \\ &= f(x^{\mathbf{a} \cdot (\mathbf{t}-1)} u) = x^{\mathbf{a} \cdot (\mathbf{t}-1)} \cdot f(u) \end{aligned}$$

Therefore we conclude that

$$\mathfrak{r}^*(f) \circ \Phi_M(u) = (\phi_{\mathbf{a}}^N)^{-1} (x^{\mathbf{a} \cdot (\mathbf{t}-1)} \cdot f(u)) = \Phi_N \circ f(u).$$

(2) Let $M \in \mathbf{Mod}_S^1$. We have to show $\Phi_M : M \rightarrow \mathfrak{r}^*(M)$ is then an isomorphism. Since both of M and $\mathfrak{r}^*(M)$ are objects in \mathbf{Mod}_S^1 , it suffices to show that each $(\Phi_M)_{\mathbf{a}} : M_{\mathbf{a}} \rightarrow \mathfrak{r}^*(M)_{\mathbf{a}}$ is an isomorphism for all \mathbf{a} with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{1}$. This is an immediate consequence of the fact that for such \mathbf{a} , the multiplication map

$$M_{\mathbf{a}} \xrightarrow{x^{\mathbf{a} \cdot (\mathbf{t}-1)}} M_{\mathbf{a} \cdot \mathbf{t}}$$

is an isomorphism since $M \in \mathbf{Mod}_S^1$. Thus we conclude that Φ_M is an isomorphism for all $M \in \mathbf{Mod}_S^1$. \square

Let \mathcal{D} denote the contravariant functor $\underline{\text{Hom}}_S(-, S) : \mathbf{Mod}_S^{\mathbb{Z}^n} \rightarrow \mathbf{Mod}_S^{\mathbb{Z}^n}$. We set $\mathcal{D}_{\mathbf{t}} := \sigma_{\mathbf{t}} \circ \mathcal{D}$. Note that $\mathcal{D}_{\mathbf{t}}$ gives a duality on $\mathbf{Mod}_S^{\mathbf{t}}$, and $\mathcal{D}_{\mathbf{1}} = \sigma_{-\mathbf{t}+\mathbf{1}} \circ \mathcal{D}_{\mathbf{t}}$ is the usual duality on $\mathbf{Mod}_S^{\mathbb{Z}^n}$ by the canonical module $S(-\mathbf{1})$ of S . The functor $\mathcal{D}_{\mathbf{t}}$, lifted up to a functor from the category of complexes in $\mathbf{Mod}_S^{\mathbf{t}}$ to itself, coincides with the one $\mathcal{D}_{\mathbf{t}}$ in [1] up to shifting and quasi-isomorphism [1, Proposition 3.6]. Moreover $\mathcal{D}_{\mathbf{t}}$ (resp. $\mathcal{D}_{\mathbf{1}}$) sends $M \in \mathbf{Mod}_S^{\mathbf{t}}$ (resp. $M \in \mathbf{Mod}_S^{\mathbf{1}}$) to an object in $\mathbf{Mod}_S^{\mathbf{t}}$ (resp. $\mathbf{Mod}_S^{\mathbf{1}}$).

Proposition 2.2. *There exists the natural isomorphisms between functors*

- (1) $\tau_0 \circ \mathcal{D}_{\mathbf{1}} \simeq \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*$ and
- (2) $\mathfrak{r}^* \circ \mathcal{D}_{\mathbf{t}} \simeq \mathcal{D}_{\mathbf{1}} \circ \mathfrak{b}^*$,

from $\mathbf{Mod}_S^{\mathbf{t}}$ to $\mathbf{Mod}_S^{\mathbf{1}}$.

Proof. (1) By Lemma 2.1, there exists the natural transformation $\Phi : \text{id}_{\mathbf{Mod}_S^{\mathbf{t}}} \Rightarrow \mathfrak{r}^*$, and hence we have the one $\tau_0 \circ \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^* \Rightarrow \tau_0 \circ \mathcal{D}_{\mathbf{1}}$, where both functors are regarded as the ones from $\mathbf{Mod}_S^{\mathbf{t}}$ to $\mathbf{Mod}_S^{\mathbb{N}^n}$. Since \mathfrak{r}^*M is a squarefree module, it follows that $\tau_0 \circ \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^* = \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*$. Consequently, the above natural transformation induces the one $\Phi' : \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^* \Rightarrow \tau_0 \circ \mathcal{D}_{\mathbf{1}}$ of functors from $\mathbf{Mod}_S^{\mathbf{t}}$ to $\mathbf{Mod}_S^{\mathbb{N}^n}$.

Note that any $M \in \mathbf{Mod}_S^{\mathbf{t}}$ has a presentation

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with F_0, F_1 free modules given by direct sums of finitely many copies of $S(-\mathbf{a})$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{t}$. Since the functors $\mathfrak{r}^*, \tau_{\mathbf{t}-\mathbf{1}}$ are exact and since $\mathcal{D}_{\mathbf{t}}$ is left exact, we have the following commutative diagram with exact rows;

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*(M) & \longrightarrow & \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*(F_0) & \longrightarrow & \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*(F_1) \\ & & \Phi'_M \downarrow & & \Psi'_{F_0} \downarrow & & \Phi'_{F_1} \downarrow \\ 0 & \longrightarrow & \tau_0 \circ \mathcal{D}_{\mathbf{1}}(M) & \longrightarrow & \tau_0 \circ \mathcal{D}_{\mathbf{1}}(F_0) & \longrightarrow & \tau_0 \circ \mathcal{D}_{\mathbf{1}}(F_1). \end{array}$$

Thus what we have to show is that $\Phi'_{S(-\mathbf{a})}$ is an isomorphism for any $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{t}$.

Let N be the cokernel of $\Phi'_{S(-\mathbf{a})} : S(-\mathbf{a}) \rightarrow S(-\sqrt{\mathbf{a}}) = \mathfrak{r}^*(S(-\mathbf{a}))$. The map $\Phi'_{S(-\mathbf{a})}$ is obviously injective, and hence the following sequence

$$0 \rightarrow S(-\mathbf{a}) \rightarrow \mathfrak{r}^*(S(-\mathbf{a})) = S(-\sqrt{\mathbf{a}}) \rightarrow N \rightarrow 0,$$

is exact. By applying $\mathcal{D}_{\mathbf{1}}$, we obtain:

$$0 \rightarrow \mathcal{D}_{\mathbf{1}}(N) \rightarrow \mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*(S(-\mathbf{a})) \rightarrow \mathcal{D}_{\mathbf{1}}(S(-\mathbf{a}))$$

As obviously $\dim N \leq n - 1$, it follows that $\mathcal{D}_{\mathbf{1}}(N) = 0$. Therefore we get the following exact sequence

$$0 \longrightarrow \mathcal{D}_{\mathbf{1}}(S(-\sqrt{\mathbf{a}})) \xrightarrow{\Phi'_{S(-\mathbf{a})}} \tau_0 \circ \mathcal{D}_{\mathbf{1}}(S(-\mathbf{a})).$$

Now $\mathcal{D}_{\mathbf{1}}(S(-\sqrt{\mathbf{a}})) \cong S(-\mathbf{1} + \sqrt{\mathbf{a}})$ and $\mathcal{D}_{\mathbf{1}}(S(-\mathbf{a})) \cong S(-\mathbf{1} + \mathbf{a})$. Easy observation shows that for $\mathbf{b} \in \mathbb{Z}^n$ with $\mathbf{b} \geq \mathbf{0}$, $S(-\mathbf{1} + \sqrt{\mathbf{a}})_{\mathbf{b}} \neq 0$ if and only if $\tau_0(S(-\mathbf{1} + \mathbf{a}))_{\mathbf{b}} \neq 0$. Therefore it follows that $\Phi'_{S(-\mathbf{a})}$ is an isomorphism.

(2) The assertion (2) can be proved in the way similar to (1). We have the natural transformation

$$\mathfrak{r}^* \circ \mathcal{D}_{\mathbf{t}} = \mathfrak{r}^* \circ \mathcal{D}_{\mathbf{1}} \circ \sigma_{\mathbf{1}-\mathbf{t}} \implies \mathfrak{r}^* \circ \mathcal{D}_{\mathbf{1}} \circ \beta^*$$

between functors from $\mathbf{Mod}_S^{\mathbf{t}}$ to $\mathbf{Mod}_S^{\mathbf{1}}$ by Lemma 2.1. Since $\mathcal{D}_{\mathbf{1}} \circ \beta^*$ sends $M \in \mathbf{Mod}_S^{\mathbf{t}}$ to an object in $\mathbf{Mod}_S^{\mathbf{1}}$, there is a natural isomorphism $\mathfrak{r}^* \circ \mathcal{D}_{\mathbf{1}} \circ \beta^* \simeq \mathcal{D}_{\mathbf{1}} \circ \beta^*$ by Lemma 2.1 again. Consequently we have the natural transformation $\Psi' : \mathfrak{r}^* \circ \mathcal{D}_{\mathbf{t}} \implies \mathcal{D}_{\mathbf{1}} \circ \beta^*$. By the argument similar as above, we have only to prove $\Psi'_{S(-\mathbf{a})}$ is an isomorphism for all \mathbf{a} with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{t}$. By the definition of Ψ , $\Psi_{S(-\mathbf{a})}$ is injective. Applying $\mathfrak{r}^* \circ \mathcal{D}_{\mathbf{1}}$ to the exact sequence

$$0 \longrightarrow \beta^*(S(-\mathbf{a})) \xrightarrow{\Psi_{S(-\mathbf{a})}} \sigma_{\mathbf{1}-\mathbf{t}}(S(-\mathbf{a})) \longrightarrow M \longrightarrow 0,$$

where M is the cokernel of $\Psi_{S(-\mathbf{a})}$, we have the exact one

$$0 \longrightarrow \mathfrak{r}^* \circ \mathcal{D}_{\mathbf{t}}(S(-\mathbf{a})) \xrightarrow{\Psi'_{S(-\mathbf{a})}} \mathcal{D}_{\mathbf{1}} \circ \beta^*(S(-\mathbf{a}))$$

We define $\mathbf{b} = (b_i)_{1 \leq i \leq n}$ by setting $b_i = 0$ if $a_i \leq t_i - 1$ and $b_i = 1$ if $a_i = t_i$. It follows that $\beta^*(S(-\mathbf{a})) \cong S(-\mathbf{b})$, and hence $\mathcal{D}_{\mathbf{1}} \circ \beta^*(S(-\mathbf{a})) \cong S(-(\mathbf{1} - \mathbf{b}))$. On the other hand, $\mathfrak{r}^* \circ \mathcal{D}_{\mathbf{t}}(S(-\mathbf{a})) \cong S(-\sqrt{\mathbf{t} - \mathbf{a}})$. Easy calculation shows $\mathbf{1} - \mathbf{b} = \sqrt{\mathbf{t} - \mathbf{a}}$, and therefore $\Psi'_{S(-\mathbf{a})}$ is an isomorphism. \square

Theorem 2.3. *The following statements hold.*

- (1) *Let ω_S be the canonical module of S . Then there are the following two isomorphisms*
 - (a) $\tau_0 \underline{\text{Ext}}_S^p(M, \omega_S) \cong \underline{\text{Ext}}_S^p(\mathfrak{r}^*(M), \omega_S)$
 - (b) $\mathfrak{r}^*(\underline{\text{Ext}}_S^p(M, S(-\mathbf{t}))) \cong \underline{\text{Ext}}_S^p(\beta^*(M), \omega_S)$*for all p and $M \in \mathbf{Mod}_S^{\mathbf{t}}$.*
- (2) *Assume I is a Cohen-Macaulay monomial ideal such that $S/I \in \mathbf{Mod}_S^{\mathbf{t}}$. Let $\omega_{S/I}, \omega_{S/\sqrt{I}}$ be the canonical modules of $S/I, S/\sqrt{I}$, respectively. Then*

$$\omega_{S/\sqrt{I}} \cong \tau_0(\omega_{S/I}) = (\omega_{S/I})_{\geq \mathbf{0}}.$$

Proof. Choose a \mathbb{Z}^n -graded minimal free resolution P_{\bullet} of M with each P_i positively \mathbf{t} -determined. By Proposition 2.2,

$$\tau_0 \underline{\text{Ext}}_S^p(M, \omega_S) \cong H^p(\tau_0 \circ \mathcal{D}_{\mathbf{1}}(P_{\bullet})) \cong H^p(\mathcal{D}_{\mathbf{1}} \circ \mathfrak{r}^*(P_{\bullet})) \cong \underline{\text{Ext}}^p(\mathfrak{r}^*(M), \omega_S),$$

since $\mathfrak{r}^*(P_{\bullet})$ is a free resolution of $\mathfrak{r}^*(M)$.

The assertion (2) follows from (1) and Theorem 1.12. \square

Remark 2.4. In [4, Corollary 2.3], Herzog, Takayama, and Terai proved

$$H_{\mathfrak{m}}^i(S/I)_{\mathbf{a}} \cong H_{\mathfrak{m}}^i(S/\sqrt{I})_{\mathbf{a}}$$

for any monomial ideal I and all i and $\mathbf{a} \leq \mathbf{0}$. In (1) of Theorem 2.3, taking the graded Matlis duality, we obtain the generalization of this result, which also implies that there exists the isomorphism

$$H_{\mathfrak{m}}^i(S/I)_{\leq \mathbf{0}} \cong H_{\mathfrak{m}}^i(S/\sqrt{I})$$

as \mathbb{Z}^n -graded S -modules. Here, for any $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$, $M_{\leq \mathbf{0}}$ denotes the residue of M by its \mathbb{Z}^n -graded submodule generated by the homogeneous elements whose degree is not less than or equal to $\mathbf{0}$.

As an immediate consequence of the above theorem we get the following corollary that generalizes a result of [4].

Corollary 2.5. *Let M be a positively \mathbf{t} -determined S -module with $\dim M = \dim \mathfrak{r}^* M$. Then:*

- (1) *M is generalized Cohen-Macaulay if and only if so is $\mathfrak{r}^* M$.*
- (2) *If M is Buchsbaum, then $\mathfrak{r}^* M$ is Buchsbaum.*

Proof. (1) M is generalized Cohen-Macaulay if and only if $\underline{\mathrm{Ext}}_S^i(M, \omega_S)$ has finite length for any $i \neq n - d$, where $d = \dim M$. This is equivalent to say that $\tau_0 \underline{\mathrm{Ext}}_S^i(M, \omega_S)$ has finite length for $i \neq n - d$. Therefore, since M and $\mathfrak{r}^* M$ have the same dimension, the desired statement follows by Corollary 2.3.

(2) If M is Buchsbaum, then M is generalized Cohen-Macaulay [2], thus $\mathfrak{r}^* M$ is generalized Cohen-Macaulay. By [9, Corollary 2.7], it follows that $\mathfrak{r}^* M$ is Buchsbaum. \square

3. THE \mathfrak{r}^* FUNCTOR AND ALEXANDER DUALITY

Recall that there exists the duality $\mathcal{A}_{\mathbf{t}}$ on $\mathbf{Mod}_S^{\mathbf{t}}$, called Alexander duality, defined by Miller [6]. In the case, $\mathbf{t} = \mathbf{1}$, Römer also defines independently in [8]. Let E denote the injective hull of K and $p_{\mathbf{t}} : \mathbb{N}^n \rightarrow \mathbb{Z}^n$ the map defined by $p_{\mathbf{t}}(\mathbf{a}) := (p_{t_1}(a_1), \dots, p_{t_n}(a_n))$, where

$$p_{t_i}(a_i) := \begin{cases} a_i & \text{if } 0 \leq a_i < t_i, \\ t_i & \text{if } a_i \geq t_i. \end{cases}$$

The functor $\mathcal{A}_{\mathbf{t}}$ is given by the formula

$$\mathcal{A}_{\mathbf{t}}(M) = p_{\mathbf{t}}^* \underline{\mathrm{Hom}}_S(M(\mathbf{t}), E).$$

If we set \mathbf{D} to be the functor $\underline{\mathrm{Hom}}_K(-, K)$, we have the natural isomorphism $\mathcal{A}_{\mathbf{t}} \cong p_{\mathbf{t}}^* \circ \sigma_{\mathbf{t}} \circ \mathbf{D}$.

The following is essentially proved by Miller in [6].

Proposition 3.1. *There exists a natural isomorphism of functors from $\mathbf{Mod}_S^{\mathbf{t}}$ to itself*

$$\mathcal{A}_{\mathbf{1}} \circ \mathfrak{r}^* \simeq \mathfrak{r}^* \circ \mathcal{A}_{\mathbf{t}}.$$

Proof. Let $M \in \mathbf{Mod}_S^{\mathbf{t}}$ and $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{a} \geq \mathbf{0}$.

$$\begin{aligned} \mathcal{A}_{\mathbf{1}} \circ \mathfrak{r}^*(M)_{\mathbf{a}} &= \underline{\mathrm{Hom}}_K(\mathfrak{r}^*(M), K)_{p_{\mathbf{1}}(\mathbf{a}) - \mathbf{1}} = \underline{\mathrm{Hom}}_K(M_{\mathfrak{r}(-p_{\mathbf{1}}(\mathbf{a}) + \mathbf{1})}, K), \\ \mathfrak{r}^* \circ \mathcal{A}_{\mathbf{t}}(M)_{\mathbf{a}} &= \underline{\mathrm{Hom}}_K(M, K)_{p_{\mathbf{t}}(\mathfrak{r}(\mathbf{a})) - \mathbf{t}} = \underline{\mathrm{Hom}}_K(M_{-p_{\mathbf{t}}(\mathfrak{r}(\mathbf{a})) + \mathbf{t}}, K). \end{aligned}$$

It is easy to verify that $\mathfrak{r}(-p_{\mathbf{1}}(\mathbf{a}) + \mathbf{1}) = -p_{\mathbf{t}}(\mathfrak{r}(\mathbf{a})) + \mathbf{t}$. Hence $\mathcal{A}_{\mathbf{1}} \circ \mathfrak{r}^*(M) \cong \mathfrak{r}^* \circ \mathcal{A}_{\mathbf{t}}(M)$ as \mathbb{Z}^n -graded K -vector spaces. By a routine calculation, we can show that this isomorphism is S -linear and natural. \square

4. RELATION TO THE AUSLANDER-REITEN TRANSLATE

Let \leq be the order on \mathbb{Z}^n defined as follows:

$$\mathbf{a} \leq \mathbf{b} \iff a_i \leq b_i$$

for all i . With the order induced by $<$, the set $P_{\mathbf{t}} := \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{0} \leq \mathbf{a} \leq \mathbf{t}\}$ becomes a poset. Let A denote the incidence algebra of $P_{\mathbf{t}}$ over K . It is well-known that the category $\mathbf{Mod}_S^{\mathbf{t}}$ is equivalent to the one mod A consisting of finite-dimensional left A -modules ([1, 3.5] and [10, Proposition 4.3]). Since A is a finite-dimensional A -algebra of finite global dimension, as is shown in [3, 3.6] by Happel, the bounded derived category $D^b(\text{mod } A)$ of mod A has Auslander-Reiten triangles. Let $K^b(\text{Proj } A)$ (resp. $K^b(\text{Inj } A)$) be the bounded homotopy category of complexes of finite-dimensional projective (resp. injective) left modules. According to Happel's proof, through the equivalence $K^b(\text{Proj } A) \cong D^b(\text{mod } A) \cong K^b(\text{Inj } A)$ of triangulated categories, the Auslander-Reiten translate (see [5, Definition 1.2] for its definition) is then given by $T^{-1} \circ v$, where T is the usual translation functor and v is the equivalence $K^b(\text{Proj } A) \cong K^b(\text{Inj } A)$ of triangulated categories induced by the Nakayama functor, i.e., $\text{Hom}_K(\text{Hom}_A(-, A), K)$. On the other hand v coincides with $\mathcal{A}_{\mathbf{t}} \circ \mathcal{D}_{\mathbf{t}}$ through the equivalence $\mathbf{Mod}_S^{\mathbf{t}} \cong \text{mod } A$. In this sense, $\mathcal{A}_{\mathbf{t}} \circ \mathcal{D}_{\mathbf{t}}$ represents the Auslander-Reiten translate of $D^b(\text{mod } A)$ (see [1, 3.3 and 3.5] for details).

In this section, we discuss the relation between \mathbf{r}^* and $\mathcal{A}_{\mathbf{t}} \circ \mathcal{D}_{\mathbf{t}}$. For this, we need to define a new functor. For $\mathbf{a} \in \mathbb{Z}^n$, let $\tau^{\mathbf{a}}$ be the functor from $\mathbf{Mod}_S^{\mathbb{Z}^n}$ to $\mathbf{Mod}_S^{\mathbb{Z}^n}$ defined as follows: for any $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$, we set

$$\tau_{\mathbf{a}}(M) := M / (S \cdot \bigoplus_{\mathbf{b} \leq \mathbf{a}} M_{\mathbf{a}}),$$

and for a morphism $f : M \rightarrow N$ in $\mathbf{Mod}_S^{\mathbb{Z}^n}$, we assign, $\tau_{\mathbf{a}}(f)$, the natural homomorphism induced by f . Clearly $\tau_{\mathbf{a}}$ is additive and exact.

Lemma 4.1. *There are the following natural isomorphisms of functors: for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$,*

- (1) $\sigma_{\mathbf{a}} \circ \tau^{\mathbf{b}} \simeq \tau^{\mathbf{a}+\mathbf{b}} \circ \sigma_{\mathbf{b}}$,
- (2) $\mathbf{D} \circ \sigma_{\mathbf{a}} \simeq \sigma_{-\mathbf{a}} \circ \mathbf{D}$, and
- (3) $\mathbf{D} \circ \tau_{\mathbf{a}} \simeq \tau^{\mathbf{a}} \circ \mathbf{D}$.

Proof. By comparing each degree \mathbf{c} component, for each $M \in \mathbf{Mod}_S^{\mathbb{Z}^n}$, we obtain an isomorphism, as \mathbb{Z}^n -graded K -vector spaces, between two modules given by applying each of two functors. An easy calculation shows these maps indeed defines the desired natural isomorphisms. \square

Proposition 4.2. *There exists the following two natural isomorphisms of functors*

- (1) $\mathcal{A}_1 \circ \mathcal{D}_1 \circ \mathbf{r}^* \simeq p_1^* \circ \mathcal{A}_{\mathbf{t}} \circ \mathcal{D}_{\mathbf{t}}$
- (2) $\mathbf{r}^* \circ \mathcal{A}_{\mathbf{t}} \circ \mathcal{D}_{\mathbf{t}} \simeq \mathcal{A}_1 \circ \mathcal{D}_1 \circ \beta^*$

from $\mathbf{Mod}_S^{\mathbf{t}}$ to \mathbf{Mod}_S^1 .

Proof. The second natural isomorphism is a direct consequence of Propositions 2.2 and 3.1. We will show the first. It follows from Proposition 2.2 and Lemma 4.1 that

$$\begin{aligned}
\mathcal{A}_1 \circ \mathcal{D}_1 \circ \mathfrak{r}^* &\simeq \mathcal{A}_1 \circ \tau_0 \circ \mathcal{D}_1 \\
&\simeq p_1^* \circ \sigma_1 \circ \mathbf{D} \circ \tau_0 \circ \sigma_{1-t} \circ \mathcal{D}_t \\
&\simeq p_1^* \circ \sigma_1 \circ \tau^0 \circ \sigma_{t-1} \circ \mathbf{D} \circ \mathcal{D}_t \\
&\simeq p_1^* \circ \tau^1 \circ \sigma_t \circ \mathbf{D} \circ \mathcal{D}_t.
\end{aligned}$$

By the definition, it is easy to verify that $p_1^* \circ \tau^1 = p_1^*$. Moreover $p_t \circ p_1 = p_1$ implies $p_1^* \circ p_t^* = p_1^*$. Thus it follows that

$$\mathcal{A}_1 \circ \mathcal{D}_1 \circ \mathfrak{r}^* \simeq p_1^* \circ \tau^1 \circ \sigma_t \circ \mathbf{D} \circ \mathcal{D}_t \simeq p_1^* \circ \mathcal{A}_t \circ \mathcal{D}_t.$$

□

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